A note on the supplementary variables in spin-measuring equipments in the EPR-Bell experiment

Won Young Hwang * and In Gyu Koh
Department of Physics, KAIST, Yusung, Taejon, Korea

Yeong Deok Han †
Department of Physics, Woosuk University, Hujeong, Samrye, Wanju, Cheonbuk, Korea

We discuss supplementary (or hidden) variables in spin-measuring equipments in EPR-Bell experiment. This theme was considered in a Bell's later work. We generalize it. First, we show why the original supplementary variable λ is not to be regarded to include supplementary variables in spin-measuring equipments (why supplementary variables should be introduced additionally in spin-measuring equipments) Next, we show the followings. When the supplementary variables introduced in spin-measuring equipments have local correlations, the Bell inequality is recovered. On the other hand, when they have nonlocal correlations, the Bell inequality is not recovered. This fact is in accord with the fact that the Bell inequality is derived for local realistic models.

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I. INTRODUCTION

Since the discovery of quantum mechanics, there have been many controversies(1) concerning whether quantum mechanics is compatible with realism. In the Copenhagen interpretation(1,2) of quantum mechanics, it was advocated that the world is no more compatible with realism and thus it is useless to make efforts to find pictures of what are happening in the world. The Copenhagen interpretation has been accepted by most physicists. However, some realistic interpretations (or realistic models) of quantum mechanics have been discovered(3,4,5). These realistic interpretations have nonlocal characters, which seem to be awkward to physicists acquainted with local realism that has been very successful for a long time in natural sciences. Thus it is an important problem whether *local* realistic interpretation of quantum mechanics is also possible (in other words whether quantum mechanical predictions can be reproduced by local realistic models). Bell(6,9) gave a surprising answer to this question; the local realism cannot coexist with quantum mechanics (The Bell theorem). This was shown by the fact that an inequality (the Bell inequality) that must be satisfied by all local realistic models is violated by quantum mechanics. In order to know which one is right, experiments have to be done. In spite of real experiments done(10), there still remain some controversies concerning the interpretations of the results, mainly concerning the detection-loophole(9,11).

By the way, in (local or nonlocal) realistic models for quantum mechanics, supplementary (or hidden) variable λ should be introduced to systems in order to explain intrinsic statistical character of quantum mechanics. In Bell's original paper(6), the supplementary variable λ corresponds to only the state of particle pairs from source of the EPR (Einstein-Podolsky-Rosen)-Bell experiment(9,12). We will show this fact in section 3. In a Bell's later work(7), the conditions are generalized: the supplementary variables are introduced also to the spin-measuring equipments of the EPR-Bell experiment and deterministic one is relaxed to probabilistic one. Bell showed(7) that Bell inequality is recovered even in those cases. The purpose of this note is to make the recovering process more general and explicit.

II. THE BELL INEQUALITY

Bell(6,9) showed that it is impossible for any local realistic models to reproduce all the predictions of quantum mechanics for EPR experiment(9,12). The experiment is described as follows. A source emits pairs of spin-$\frac{1}{2}$ particles, in the singlet state $\frac{1}{\sqrt{2}}(|z+\rangle|z-\rangle - |z-\rangle|z+\rangle)$. After particles have separated, one performs correlated measurements of their spin components along arbitrary directions a and b (Fig.1). Each measurement can yield two results, ± 1. For the singlet state, quantum mechanics predicts some correlations between such measurements on the two particles.

*e-mail: hwang@chep6.kaist.ac.kr
†e-mail: ydhan@core.woosuk.ac.kr

We denote by $p_{\pm\pm}(a,b)$ the probabilities of obtaining the result ± 1 along a at one place A and ± 1 along b at the other place B. The quantity

$$E(a,b) = p_{++}(a,b) + p_{--}(a,b) - p_{+-}(a,b) - p_{-+}(a,b) \tag{1}$$

is the correlation function of measurements of two particles.

In (deterministic) realistic models, it is assumed that the outcome of measurements of spin is determined by some supplementary (or hidden) variable λ and by the directions a and b of spin-measuring equipments. The supplementary variable λ is a random variable; the probability distribution of which is given by some function $\rho_{ab}(\lambda)$ having the normalization property

$$\int \rho_{ab}(\lambda)\ d\lambda = 1, \tag{2}$$

and depending on a,b and on the state ψ of the pair of particles. The outcomes of spin-measurements are assumed to be determined by λ and by direction of each corresponding spin-measurement;

$$S_A(a) = f(a,\lambda), \qquad S_B(b) = g(b,\lambda). \qquad \text{the locality condition} \tag{3}$$

The functions f and g depend on the particular properties of measuring equipments and the range of which are $\{\pm 1\}$.

The combination of correlation functions

$$S = E(a,b) + E(a,b') + E(a',b) - E(a',b') \tag{4}$$

is constrained by these properties of local realistic models. $E(a,b)$ in the case of local realistic models is given by Eq.(1) and Eq.(3);

$$E(a,b) = \int f(a,\lambda) g(b,\lambda) \rho_{ab}(\lambda) d\lambda \tag{5}$$

With these equations,

$$\begin{aligned} \mid S \mid &= \mid E(a,b) + E(a,b') + E(a',b) - E(a',b') \mid, \\ &= \mid \int f(a,\lambda) g(b,\lambda) \rho_{ab}(\lambda) d\lambda + \int f(a,\lambda) g(b',\lambda) \rho_{a',b}(\lambda) d\lambda \\ &\quad + \int f(a',\lambda) g(b,\lambda) \rho_{a,b'}(\lambda) d\lambda - \int f(a',\lambda) g(b',\lambda) \rho_{a',b'}(\lambda) d\lambda \mid. \end{aligned} \tag{6}$$

If λ corresponds to only particle pairs from source, it can be justified from locality condition that the probability distribution $\rho_{ab}(\lambda)$ is independent upon a and b, the directions of spin-component being measured at A and B respectively,

$$\rho_{ab}(\lambda) = \rho_{a,b'}(\lambda) = \rho_{a',b}(\lambda) = \rho_{a',b'}(\lambda) \equiv \rho(\lambda). \tag{7}$$

Thus,

$$\begin{aligned} |S| &= \mid \int [f(a,\lambda)\{g(b,\lambda) + g(b',\lambda)\} + f(a',\lambda)\{g(b,\lambda) - g(b',\lambda)\}] \rho(\lambda) d\lambda \mid, \\ &\leq \int \{|f(a,\lambda)||g(b,\lambda) + g(b',\lambda)| + |f(a',\lambda)||g(b,\lambda) - g(b',\lambda)|\} \rho(\lambda) d\lambda, \\ &\leq \int \{|g(b,\lambda) + g(b',\lambda)| + |g(b,\lambda) - g(b',\lambda)|\} \rho(\lambda) d\lambda, \\ &\leq \int 2\rho(\lambda) d\lambda = 2. \end{aligned} \tag{8}$$

Eq.(8) is derived from the facts that $|f| \leq 1$ and $|g| \leq 1$. This is the Bell inequality,

$$\mid E(a,b) + E(a,b') + E(a',b) - E(a',b') \mid \leq 2, \tag{9}$$

violated by quantum mechanics at certain cases.

We might think over broader classes of local realistic models where the assumption (3) is a little relaxed. These are 'the stochastic local realistic model[7,13,14] ' and 'the contextual local realistic model[9]'. In the former case, determinism is relaxed; the probability that spin-measurements along a direction a (b) at A (at B) for a certain λ will give an outcome m (n) ($m, n = \pm 1$) is given $p(m|a, \lambda)$ ($p(n|b, \lambda)$). In this case, after some algebraic manipulations we obtain, for example,

$$E(a, b) = \int \bar{f}(a, \lambda)\bar{g}(b, \lambda)\rho_{ab}(\lambda)d\lambda,$$
$$\text{where} \qquad \bar{f}(a, \lambda) = \sum_m mp(m|a, \lambda), \qquad \bar{g}(b, \lambda) = \sum_n np(n|b, \lambda). \tag{10}$$

,which is the same form as that of Eq.(5). Thus, by simply replacing $f(a, \lambda)$ and $g(b, \lambda)$ by $\bar{f}(a, \lambda)$ and $\bar{g}(b, \lambda)$, we can generalize all proofs in this letter to the case of stochastic local realistic models.

In the latter case, contextual interactions within light-cone is permitted; outcomes of spin-measurements depend also on direction of spin-measurements on the other side, if the two events are not space-likely separated,

$$S_A(a) = f(a, b, \lambda), \qquad S_B(b) = g(a, b, \lambda). \tag{11}$$

In this case, to recover the Bell inequality, Eq.(11) is reduced to Eq.(3) by space-likely separating the events of detections of spin at A from those at B.

III. SUPPLEMENTARY (OR HIDDEN) VARIABLES IN SPIN MEASURING EQUIPMENTS

If spin-measuring equipments are composed of similar ones as the detected particles, it is natural to introduce additional supplementary variables λ_i ($i = a, a', b, b'$) (Fig.1).

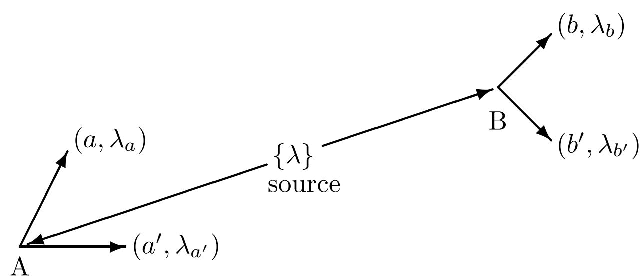

FIG. 1. λ_i ($i = a, a', b, b'$) are supplementary variables in spin-measuring equipments.

Then, since λ_i as well as λ take parts in measuring process, the outcome of spin-measurements are determined by λ_i as well as λ and directions of spin-measuring equipments,

$$\begin{aligned} S_A(a) &= f(a, \lambda, \lambda_a) \qquad & S_B(a) &= g(b, \lambda, \lambda_b) \\ S_A(a') &= f(a', \lambda, \lambda_{a'}) \qquad & S_B(b) &= g(b', \lambda, \lambda_{b'}) \end{aligned} \tag{12}$$

One may argue that we do not need to additionally introduce these variables λ_i, because the original variable λ can be regarded to include all these variables, since there is no specification that λ corresponds to only source particle pairs. We may answer to this argument as follows. As shown in section 2 the independence of $\rho_{ab}(\lambda)$ on a and b (Eq.(7)) was used for derivation of Bell inequality. And this independence was justified by locality only when λ corresponds to only source particles which are space-likely separated from the measuring process at each side. Thus, if λ corresponds to spin-measuring equipments as well as to source particles, then the independence condition (Eq.(7)) is not given. So, Bell inequality is not obtained immediately without some operations which will be described in the following. The above demonstration is applied also to the slightly different version[15,13,14] of Bell inequality (where the locality condition (Eq.(3)) is replaced by factorizability (Eq.(2′) of Ref.(15)), because the independency condition (Eq.(7)) is also assumed implicitly in Ref.(15).

IV. RECOVERING THE BELL INEQUALITY

The probability distributions $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ ($p=a,a' \quad q=b,b'$) of λ corresponding to particles pairs from source and λ_i ($i=a,a',b,b'$) corresponding to spin-measuring equipments are not independent upon a and b. With these general distributions $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$, we obviously cannot recover Bell inequality. However, the locality will constraint the $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ in some ways. The first possibility is that they factorize to each part, for example,

$$\rho_{ab}(\lambda,\lambda_a,\lambda_b)=\rho_{ab}(\lambda)\rho_{ab}(\lambda_a)\rho_{ab}(\lambda_b). \tag{13}$$

This factorization is a physically plausible assumption, since this is equivalent to assuming that source and measuring equipments of each side are independent upon each other. By locality we have, for example,

$$\rho_{ab}(\lambda)\rho_{ab}(\lambda_a)\rho_{ab}(\lambda_b)=\rho(\lambda)\rho_a(\lambda_a)\rho_b(\lambda_b). \tag{14}$$

Then we have, for example,

$$\begin{aligned}E(a,b)&=\int\rho(\lambda)\rho_a(\lambda_a)\rho_b(\lambda_b)f(a,\lambda,\lambda_a)g(b,\lambda,\lambda_b)d\lambda d\lambda_a d\lambda_b\\&=\int[\int f(a,\lambda,\lambda_a)\rho_a(\lambda_a)d\lambda_a][\int g(a,\lambda,\lambda_b)\rho_b(\lambda_b)d\lambda_b]\rho(\lambda)d\lambda\\&=\int\bar{F}(a,\lambda)\bar{G}(b,\lambda)\rho(\lambda)d\lambda, \qquad (15)\\ \text{where}\quad \bar{F}(a,\lambda)&\equiv\int f(a,\lambda,\lambda_a)\rho_a(\lambda_a)d\lambda_a,\qquad \bar{G}(b,\lambda)\equiv\int g(a,\lambda,\lambda_b)\rho_b(\lambda_b)d\lambda_b\end{aligned}$$

which is the same form as that of Eq.(5) so that we can recover Bell inequality. Similarity between Eq.(10) and Eq.(15) can be used in order that deterministic local realistic models with some extra variables λ_i in spin-measuring equipments (Eq.(15)) emulate stochastic local realistic models (Eq.(10)).

The next possibility we consider is the case where there exist a joint probability distribution $\rho(\lambda,\lambda_a,\lambda_{a'},\lambda_b,\lambda_{b'})$ which returns $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ as marginal probability distributions, for example,

$$\rho_{ab}(\lambda,\lambda_a,\lambda_b)=\int\rho(\lambda,\lambda_a,\lambda_{a'},\lambda_b,\lambda_{b'})d\lambda_{a'}d\lambda_{b'}. \tag{16}$$

In this case we have, for example,

$$E(a,b)=\int f(a,\lambda,\lambda_a)g(b,\lambda,\lambda_b)\rho(\lambda,\lambda_a,\lambda_{a'},\lambda_b,\lambda_{b'})d\lambda d\lambda_a d\lambda_{a'}d\lambda_b d\lambda_{b'}. \tag{17}$$

We define,

$$\lambda\otimes\lambda_a\otimes\lambda_{a'}\otimes\lambda_b\otimes\lambda_{b'}\equiv\tilde{\lambda}. \tag{18}$$

And we regard $f(a,\lambda,\lambda_a)$ ($g(b,\lambda,\lambda_b)$) as a function of a and $\tilde{\lambda}$ (b and $\tilde{\lambda}$)

$$f(a,\lambda,\lambda_a)\equiv f(a,\tilde{\lambda}),\qquad g(b,\lambda,\lambda_b)\equiv g(b,\tilde{\lambda}). \tag{19}$$

Then we have, for example,

$$E(a,b)=\int f(a,\tilde{\lambda})g(b,\tilde{\lambda})\rho(\tilde{\lambda})d\tilde{\lambda} \tag{20}$$

which is the same form as that of Eq.(5). Therefore, we can recover Bell inequality.

Let us now discuss about the physical meanings of the existence of a joint distribution $\rho(\tilde{\lambda})$ which returns $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ as marginal distributions. The existence of such a joint distribution $\rho(\tilde{\lambda})$ is equivalent to the existence of deterministic local realistic models which reproduce $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$[13]. That is, if $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ can be obtained from a single joint distribution as marginals then these $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ can be reproduced by some local realistic models, and if $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ cannot be obtained from a single joint distribution as marginals then these $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ cannot be reproduced by any local realistic models. Thus we may call $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ which have such a joint distribution *local correlations* of λ,λ_p and λ_q, and call $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ which does not have such a joint distribution *nonlocal correlations* of λ,λ_p and λ_q. On the other hand, we could show that Bell inequality can be recovered (Eq.(20)) in the case of these local correlations of λ,λ_p and λ_q, while we could not show that Bell inequality can be recovered in the case of nonlocal correlations of λ,λ_p and λ_q. This fact is in accord with the fact that Bell inequality is obtained for *local* realistic models.

V. DISCUSSION AND CONCLUSION

It is obvious that Bell inequality cannot be recovered without some constraints on $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ ($p=a,a' \quad q=b,b'$) from locality. In previous section, we could show that Bell inequality can be recovered in two cases. (i) $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ are factorized (Eq.(14)), (ii) there exists a joint distribution $\rho(\tilde{\lambda})$ which returns $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ as marginal distributions (Eq.(16)). By the way, case(i) is a subset of case(ii), because Eq.(14) can be reproduced by

$$\rho(\tilde{\lambda}) = \rho(\lambda)\rho_a(\lambda_a)\rho_{a'}(\lambda_{a'})\rho_b(\lambda_b)\rho_{b'}(\lambda_{b'}). \tag{21}$$

In the case of (i), the distribution of λ_p and λ_q are independent each other. In the case of (ii), the distribution of λ_p and λ_q may have correlations, unless they are nonlocal one. By the way, in the Bell's later work(7) the independency case (condition (i)) is considered. In this connection, we generalized the Bell's later work to the case of condition (ii).

Summing up the above results, we could say that if we ignore λ_i ($i=a,a',b,b'$) the three following ones are reduced to an identical one. (a) stochastic models without λ_i(15,13,14) (b) deterministic models with λ_i (which are correlated (condition (i)) or not (condition (ii)). (c) stochastic models with λ_i(7).

With nonlocal correlations $\rho_{pq}(\lambda,\lambda_p,\lambda_q)$ of λ and λ_i , Bell inequality was not obtained. In other words, if the distributions of λ and λ_i are nonlocal (or do not have a joint distribution) then they can give outcomes that violate the Bell inequality (Eq.(9)) for which λ_i acted as measuring equipments. Recently, it was shown that violation of the Bell inequality does not necessarily mean that all the source particle pairs are nonlocal ones(16), that is, the nonlocality can be ascribed to some subsets of ensemble of particle pairs. Here we have shown that the nonlocality might be ascribed to spin-measuring equipments.

In summary, the original supplementary variable λ is not to be regarded to include supplementary variables in spin-measuring equipments as shown in section 3. Thus supplementary variables should be introduced additionally in spin-measuring equipments. When the supplementary variables introduced in spin-measuring equipments have local correlations, the Bell inequality is recovered. On the other hand, when they have nonlocal correlations, the Bell inequality is not recovered. This fact is in accord with the fact that the Bell inequality is derived for local realistic models.

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